

Solving quadratic equations by factorisation

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose products is ac .
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

Examples

Example 1 Solve $5x^2 = 15x$

$5x^2 = 15x$ $5x^2 - 15x = 0$ $5x(x - 3) = 0$ <p>So $5x = 0$ or $(x - 3) = 0$</p> <p>Therefore $x = 0$ or $x = 3$</p>	<ol style="list-style-type: none"> 1 Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by x as this would lose the solution $x = 0$. 2 Factorise the quadratic equation. $5x$ is a common factor. 3 When two values multiply to make zero, at least one of the values must be zero. 4 Solve these two equations.
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Example 2 Solve $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$ $b = 7, ac = 12$ $x^2 + 4x + 3x + 12 = 0$ $x(x + 4) + 3(x + 4) = 0$ $(x + 4)(x + 3) = 0$ <p>So $(x + 4) = 0$ or $(x + 3) = 0$</p> <p>Therefore $x = -4$ or $x = -3$</p>	<ol style="list-style-type: none"> 1 Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3) 2 Rewrite the b term ($7x$) using these two factors. 3 Factorise the first two terms and the last two terms. 4 $(x + 4)$ is a factor of both terms. 5 When two values multiply to make zero, at least one of the values must be zero. 6 Solve these two equations.
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Example 3 Solve $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ <p>So $(3x + 4) = 0$ or $(3x - 4) = 0$</p> $x = -\frac{4}{3} \text{ or } x = \frac{4}{3}$	<ol style="list-style-type: none"> 1 Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$. 2 When two values multiply to make zero, at least one of the values must be zero. 3 Solve these two equations.
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Example 4 Solve $2x^2 - 5x - 12 = 0$

$b = -5, ac = -24$ <p>So $2x^2 - 8x + 3x - 12 = 0$</p> $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ <p>So $(x - 4) = 0$ or $(2x + 3) = 0$</p> $x = 4 \text{ or } x = -\frac{3}{2}$	<ol style="list-style-type: none"> 1 Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. (-8 and 3) 2 Rewrite the b term ($-5x$) using these two factors. 3 Factorise the first two terms and the last two terms. 4 $(x - 4)$ is a factor of both terms. 5 When two values multiply to make zero, at least one of the values must be zero. 6 Solve these two equations.
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Practice

1 Solve

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| <p>a $6x^2 + 4x = 0$</p> <p>c $x^2 + 7x + 10 = 0$</p> <p>e $x^2 - 3x - 4 = 0$</p> <p>g $x^2 - 10x + 24 = 0$</p> <p>i $x^2 + 3x - 28 = 0$</p> <p>k $2x^2 - 7x - 4 = 0$</p> | <p>b $28x^2 - 21x = 0$</p> <p>d $x^2 - 5x + 6 = 0$</p> <p>f $x^2 + 3x - 10 = 0$</p> <p>h $x^2 - 36 = 0$</p> <p>j $x^2 - 6x + 9 = 0$</p> <p>l $3x^2 - 13x - 10 = 0$</p> |
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2 Solve

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| <p>a $x^2 - 3x = 10$</p> <p>c $x^2 + 5x = 24$</p> <p>e $x(x + 2) = 2x + 25$</p> <p>g $x(3x + 1) = x^2 + 15$</p> | <p>b $x^2 - 3 = 2x$</p> <p>d $x^2 - 42 = x$</p> <p>f $x^2 - 30 = 3x - 2$</p> <p>h $3x(x - 1) = 2(x + 1)$</p> |
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Hint

Get all terms onto one side of the equation.

Solving quadratic equations by completing the square

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

Examples

Example 5 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$x^2 + 6x + 4 = 0$ $(x + 3)^2 - 9 + 4 = 0$ $(x + 3)^2 - 5 = 0$ $(x + 3)^2 = 5$ $x + 3 = \pm\sqrt{5}$ $x = \pm\sqrt{5} - 3$ <p>So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$</p>	<ol style="list-style-type: none"> Write $x^2 + bx + c = 0$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0$ Simplify. Rearrange the equation to work out x. First, add 5 to both sides. Square root both sides. Remember that the square root of a value gives two answers. Subtract 3 from both sides to solve the equation. Write down both solutions.
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Example 6 Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$ $2\left(x^2 - \frac{7}{2}x\right) + 4 = 0$ $2\left[\left(x - \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right] + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{49}{8} + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{17}{8} = 0$	<ol style="list-style-type: none"> Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$ Now complete the square by writing $x^2 - \frac{7}{2}x$ in the form $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$ Expand the square brackets. Simplify. <p style="text-align: right;"><i>(continued on next page)</i></p>
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$2\left(x - \frac{7}{4}\right)^2 = \frac{17}{8}$ $\left(x - \frac{7}{4}\right)^2 = \frac{17}{16}$ $x - \frac{7}{4} = \pm \frac{\sqrt{17}}{4}$ $x = \pm \frac{\sqrt{17}}{4} + \frac{7}{4}$ <p>So $x = \frac{7}{4} - \frac{\sqrt{17}}{4}$ or $x = \frac{7}{4} + \frac{\sqrt{17}}{4}$</p>	<p>5 Rearrange the equation to work out x. First, add $\frac{17}{8}$ to both sides.</p> <p>6 Divide both sides by 2.</p> <p>7 Square root both sides. Remember that the square root of a value gives two answers.</p> <p>8 Add $\frac{7}{4}$ to both sides.</p> <p>9 Write down both the solutions.</p>
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Practice

3 Solve by completing the square.

a $x^2 - 4x - 3 = 0$

c $x^2 + 8x - 5 = 0$

e $2x^2 + 8x - 5 = 0$

b $x^2 - 10x + 4 = 0$

d $x^2 - 2x - 6 = 0$

f $5x^2 + 3x - 4 = 0$

4 Solve by completing the square.

a $(x - 4)(x + 2) = 5$

b $2x^2 + 6x - 7 = 0$

c $x^2 - 5x + 3 = 0$

Hint

Get all terms onto one side of the equation.

Solving quadratic equations by using the formula

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for a , b and c .

Examples

Example 7 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1, b = 6, c = 4$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{20}}{2}$ $x = \frac{-6 \pm 2\sqrt{5}}{2}$ $x = -3 \pm \sqrt{5}$ <p>So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$</p>	<ol style="list-style-type: none"> Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. Substitute $a = 1$, $b = 6$, $c = 4$ into the formula. Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2. Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$ Simplify by dividing numerator and denominator by 2. Write down both the solutions.
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Example 8 Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

$a = 3, b = -7, c = -2$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$ $x = \frac{7 \pm \sqrt{73}}{6}$ <p>So $x = \frac{7 - \sqrt{73}}{6}$ or $x = \frac{7 + \sqrt{73}}{6}$</p>	<ol style="list-style-type: none"> 1 Identify a, b and c, making sure you get the signs right and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it. 2 Substitute $a = 3$, $b = -7$, $c = -2$ into the formula. 3 Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2. 4 Write down both the solutions.
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Practice

5 Solve, giving your solutions in surd form.

a $3x^2 + 6x + 2 = 0$

b $2x^2 - 4x - 7 = 0$

6 Solve the equation $x^2 - 7x + 2 = 0$

Give your solutions in the form $\frac{a \pm \sqrt{b}}{c}$, where a , b and c are integers.

7 Solve $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

Hint

Get all terms onto one side of the equation.

Extend

8 Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

a $4x(x - 1) = 3x - 2$

b $10 = (x + 1)^2$

c $x(3x - 1) = 10$

Answers

- 1 a** $x = 0$ or $x = -\frac{2}{3}$ **b** $x = 0$ or $x = \frac{3}{4}$
c $x = -5$ or $x = -2$ **d** $x = 2$ or $x = 3$
e $x = -1$ or $x = 4$ **f** $x = -5$ or $x = 2$
g $x = 4$ or $x = 6$ **h** $x = -6$ or $x = 6$
i $x = -7$ or $x = 4$ **j** $x = 3$
k $x = -\frac{1}{2}$ or $x = 4$ **l** $x = -\frac{2}{3}$ or $x = 5$
- 2 a** $x = -2$ or $x = 5$ **b** $x = -1$ or $x = 3$
c $x = -8$ or $x = 3$ **d** $x = -6$ or $x = 7$
e $x = -5$ or $x = 5$ **f** $x = -4$ or $x = 7$
g $x = -3$ or $x = 2\frac{1}{2}$ **h** $x = -\frac{1}{3}$ or $x = 2$
- 3 a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ **b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$
c $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$ **d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$
e $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$ **f** $x = \frac{-3 + \sqrt{89}}{10}$ or $x = \frac{-3 - \sqrt{89}}{10}$
- 4 a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$ **b** $x = \frac{-3 + \sqrt{23}}{2}$ or $x = \frac{-3 - \sqrt{23}}{2}$
c $x = \frac{5 + \sqrt{13}}{2}$ or $x = \frac{5 - \sqrt{13}}{2}$
- 5 a** $x = -1 + \frac{\sqrt{3}}{3}$ or $x = -1 - \frac{\sqrt{3}}{3}$ **b** $x = 1 + \frac{3\sqrt{2}}{2}$ or $x = 1 - \frac{3\sqrt{2}}{2}$
- 6** $x = \frac{7 + \sqrt{41}}{2}$ or $x = \frac{7 - \sqrt{41}}{2}$
- 7** $x = \frac{-3 + \sqrt{89}}{20}$ or $x = \frac{-3 - \sqrt{89}}{20}$
- 8 a** $x = \frac{7 + \sqrt{17}}{8}$ or $x = \frac{7 - \sqrt{17}}{8}$
b $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$
c $x = -1\frac{2}{3}$ or $x = 2$